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1977 J. Phys. A: Math. Gen. 10 1319

(http://iopscience.iop.org/0305-4470/10/8/011)

J. Phys. A: Math. Gen., Vol. 10, No. 8, 1977. Printed in Great Britain. © 1977

Thermodynamics of quenched random spin systems, and application to the problem of phase transitions in magnetic (spin) glasses

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Received 14 February 1977, in final form 12 April 1977

Abstract. Existence with probability one of the thermodynamic limit for the free energy density of a class of classical and quantum quenched random spin systems is proved using strong laws of large numbers. When the control on the randomness is strong enough, the infinite-volume free energy so obtained is shown to be equal, with probability one, to that derived from a prescription originally due to Brout. Particular mean-field models are then studied in detail. Similar arguments are pointed out concerning the corresponding thermodynamic functions, and are then applied to the existence problem of phase transitions in quenched random systems with continuous internal symmetry groups, in particular to those models recently proposed to describe the spin glass phenomenon, like the Edwards–Anderson model and its various classical and quantum extensions.

1. Introduction

Starting from a new prescription for evaluating free energies, Brout proposed, a little less than twenty years ago (Brout 1959), a new statistical mechanical theory for a class of random systems where magnetic ions are frozen at random positions in some non-magnetic host metals. The claim was indeed that, since such random materials do not correspond to ordinary thermal equilibrium, but rather to some metastable state, the physical free energy should not be evaluated by taking the logarithm of their averaged partition function, but rather by calculating the average over all ion configurations of the conditional free energy related to a fixed configuration of impurities (quenched impurity problem). More or less convincing physical arguments were given in Brout's paper to defend that point of view but, in spite of its numerous subsequent applications, and in spite of Mazo's approach to the same problem (Mazo 1963), no rigorous probabilistic justification of such a prescription has, to the best of our knowledge, ever been proposed. And since the last few years have revealed, among solid state theorists, a renewed interest for studying such random systems, especially because of the intriguing features of the spin glass problem, we have thought that a detailed study of Brout's prescription within the framework of rigorous statistical mechanics would be necessary, both because it is, at the present time, constantly used in the spin glass problem (see the works quoted below), and because it is usually followed, to be of any practical interest in statistical physics, by rather formal procedures like the famous '$n \to 0$ trick' which has not been mathematically justified so far.

To be more precise, this paper will be organised as follows: in § 2 we prove, using strong laws of large numbers, the existence with probability one of a thermodynamical limit for the free energy density, considered as a random variable, of a wide class of classical and quantum quenched random spin systems, and we establish its relation with the infinite-volume free energy density evaluated from Brout's prescription. Both are indeed equal with probability one under some conditions which control the randomness in the models; our argument also provides an extension of considerations about the infinite-volume theory of the two-dimensional Ising–Onsager system with quenched impurities given by McCoy and Wu (McCoy 1970, McCoy and Wu 1971, McCoy 1972 and references therein.) In § 3 particular mean-field random models are studied and their connection with the Edwards–Anderson model for spin glasses (Edwards and Anderson 1975, 1976) and its various classical and quantum extensions (Fischer 1975, Sherrington and Kirkpatrick 1975, Sherrington and Southern 1975, Thouless *et al* 1977) is pointed out. In § 4 we point out that similar arguments can be developed for the thermodynamic functions, and we then apply our results to the existence problem of phase transitions in quenched random systems with continuous internal symmetry groups. The last section is concerned with some remarks, comments and open problems.

2. Thermodynamic limit for a class of quenched random spin systems

In what follows, the ν-dimensional cubic lattice will be denoted by $\mathbb{Z}^\nu$; let $\Lambda \subset \mathbb{Z}^\nu$ be a finite box whose cardinality is denoted by $|\Lambda|$. We shall be concerned with Ising–Heisenberg–Stanley spin systems defined from the Hamiltonian

$$-H_\Lambda = \sum_{r,r' \in \Lambda} J_{rr'}(\boldsymbol{S}_r \cdot \boldsymbol{S}_{r'}) + \lambda \sum_{r\in\Lambda} S_r^D \tag{2.1}$$

and with their various classical (and quantum) extensions like the ones proposed in the work by Vuillermot and Romerio (1975, especially § 4). In (2.1) we have $D \in \mathbb{N}$, $\lambda \in \mathbb{R}$ (external field); the $\boldsymbol{S}_r$ are, for any $r \in \Lambda$, unit vectors in the D-dimensional Euclidean space $\mathbb{R}^D$, more precisely elements of the unit sphere S^{D-1} whose Dth component is denoted by S_r^D; for $D=1$ the model (2.1) is, as usual, identified with a spin-$\frac{1}{2}$ Ising model. The spin–spin couplings $J_{rr'}$ are real, not necessarily identically distributed, independent random variables; we shall write $\mathbf{J}_\Lambda$, $\mathbf{J}_\Lambda^0$ and $\boldsymbol{\sigma}_\Lambda^2$ for the random matrix $(J_{rr'})_{r,r'\in\Lambda}$, the mean matrix $(J_{rr'}^0)_{r,r'\in\Lambda}$ and the variance matrix $(\sigma_{rr'}^2)_{r,r'\in\Lambda}$ respectively; these three matrices are supposed to have vanishing diagonal elements, strictly finite elements otherwise. The corresponding probability measure on $\mathbb{R}^{|\Lambda|(|\Lambda|-1)/2}$ will be denoted by $\langle\ \rangle_{\mathbf{J}_\Lambda}$; it is supposed to be absolutely continuous with respect to the Lebesgue measure, namely

$$\langle\ \rangle_{\mathbf{J}_\Lambda} = \prod_{r\neq r'\in\Lambda} \mathrm{d}J_{rr'} P_{rr'}(J_{rr'})$$

with

$$J_{rr'}^0 \equiv \langle J_{rr'}\rangle_{\mathbf{J}_\Lambda} = \int_{\mathbb{R}} \mathrm{d}J_{rr'} P_{rr'}(J_{rr'}) J_{rr'}$$

and

$$\sigma_{rr'}^2 = \langle J_{rr'}^2\rangle_{\mathbf{J}_\Lambda} - (J_{rr'}^0)^2.$$

Observe that if we choose a Gaussian random matrix $\mathbf{J}_\Lambda$ with identically distributed elements of non-negative mean, in other words

$$P_{rr'}(J_{rr'}) = P(J_{rr'}) = \frac{1}{(2\pi\sigma)^{1/2}} \exp\left(-\frac{(J_{rr'} - J^0)^2}{2\sigma^2}\right)$$

with $J^0 \geq 0$, the Hamiltonian (2.1), and its quantum version for $D = 3$, correspond to the Edwards–Anderson model for spin glasses (Edwards and Anderson 1975, 1976) and to its various extensions (Fischer 1975, Sherrington and Kirkpatrick 1975, Sherrington and Southern 1975, Thouless *et al* 1977). Such models are usually supposed to simulate the essential features regarding the existence and qualitative properties of the so called spin glass phase observed in some disordered metallic alloys like Au–Fe. Now introduce the free energy density

$$f_\Lambda(\mathbf{J}_\Lambda; \lambda) = |\Lambda|^{-1} \ln \int_{S^{(D-1)|\Lambda|}} \mathrm{d}\boldsymbol{S}_\Lambda \exp(-H_\Lambda(\mathbf{J}_\Lambda; \lambda)) \tag{2.2}$$

considered as a random variable; in (2.2), $\mathrm{d}\boldsymbol{S}_\Lambda$ denotes the usual uniform measure on $S^{(D-1)|\Lambda|} \equiv \Pi_{r\in\Lambda} S^{(D-1)}$, and the inverse temperature $\beta = (k_B T)^{-1}$ has been normalised to one. Furthermore for $a = (a_1, \ldots, a_\nu) \in \mathbb{Z}^\nu$ with $a_i > 0$ for each i, consider the box

$$\Lambda(a) = \{r \in \mathbb{Z}^\nu; 0 \leq r_i < a_i; i = 1, \ldots, \nu\}$$

and write $N_a^-(\Lambda)$ for the number of disjoint translates of $\Lambda(a)$ strictly included in Λ; we shall denote these translates by $(\Lambda_j)_{j=1}^{N_a^-(\Lambda)}$; let $N_a^+(\Lambda)$ be the number of translates of $\Lambda(a)$ whose intersection with Λ is non-empty (Ruelle 1969, especially chap. 2); by definition, $N_a^-(\Lambda) \leq N_a^+(\Lambda)$ but we shall perform the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^\nu$ in the sense of Van Hove, namely $N_a^-(\Lambda) \to +\infty$, and then $|\Lambda| \to +\infty$, and $N_a^+(\Lambda)/N_a^-(\Lambda) \to 1$ for all a. In that context the first ingredient used below to prove the existence of the thermodynamic limit is the following inequality.

Lemma 2.1. Suppose the random variable

$$M_\nu(\mathbf{J}) = \sum_{r\in\mathbb{Z}^\nu} \sup_{r'\in\mathbb{Z}^\nu} |J_{r+r',r'}| \tag{2.3}$$

exists and is finite with probability one. Then for any $\epsilon > 0$, a large enough and Λ sufficiently large in the sense of Van Hove, we have

$$\left| f_\Lambda(\mathbf{J}_\Lambda; \lambda) - \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j}(\mathbf{J}_{\Lambda_j}; \lambda) \right| \leq \epsilon \tag{2.4}$$

with probability one. (In (2.4) the symbol $\mathbf{J}_{\Lambda_j}$ denotes the submatrix of $\mathbf{J}_\Lambda$ corresponding to the random bonds in Λ_j.)

Proof. For a fixed configuration of impurities such that $M_\nu(\mathbf{J})$ is finite, (2.4) just follows from a slight modification of the original arguments by Griffiths (1964) and Ruelle (1969); such a modification is necessary to take into account the absence of any translational invariance in the interaction constants $J_{rr'}$ (see Ruelle 1972, Roos 1975). Now from a probabilistic point of view, that means that the occurrence of the event $M_\nu(\mathbf{J}) < +\infty$ implies the occurrence of (2.4); the probability of the former is therefore less than or equal to the probability of the latter; thus if $M_\nu(\mathbf{J}) < +\infty$ occurs with probability one, (2.4) also occurs with probability one. This proves the lemma.

Remarks. Technically speaking, an inequality like (2.4) is always derived first for interactions of strictly finite range. The general case of interactions such that $M_\nu(\mathbf{J}) < +\infty$ is then obtained by a continuity argument (see the works quoted above). Now to have $M_\nu(\mathbf{J}) < +\infty$ with probability one, it is sufficient that, in a few particular cases,

$$\sum_{r\in\mathbf{Z}^\nu} \langle \sup_{r'\in\mathbf{Z}^\nu} |J_{r+r',r'}|\rangle < +\infty$$

according to Beppo Levi's theorem (Riesz and Nagy 1968). Other conditions involving the variances can be proposed as well. Very roughly speaking then one can say that the validity of (2.4) with probability one is only established providing there is a sufficiently fast decrease of the means and of the variances of the random spin couplings at large distances. On the other hand, observe that the inequality (2.4) does not prove the existence of the thermodynamic limit for $f_\Lambda(\mathbf{J}_\Lambda; \lambda)$ yet, except if $f_{\Lambda_j} = f_{\Lambda(a)}$ with probability one for each j; that particular situation corresponds to the case where $J_{rr'} = J(r-r')$; with probability one then the convergence of $M_\nu(\mathbf{J}) = \Sigma_{r\in\mathbf{Z}^\nu}|J(r)|$ is ensured by the convergence of $\Sigma_{r\in\mathbf{Z}^\nu}\langle|J(r)|\rangle_{\mathbf{J}_\Lambda}$ and of $\Sigma_{r\in\mathbf{Z}^\nu}\langle(|J(r)| - \langle|J(r)|\rangle_{\mathbf{J}_\Lambda})^2\rangle_{\mathbf{J}_\Lambda}$ according to our preceding remark. In the general case we are faced with however, the situation is not so simple any more, and since we still have to estimate the absolute value of the difference between f_Λ and $f_{\Lambda(a)}$ to get the Van Hove limit of f_Λ we first have to ensure the existence of

$$g_{\Lambda(a)} \equiv \lim_{N_a^-(\Lambda)\to+\infty} \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j}$$

for each a with probability one, and then the one of $\lim_{a\to+\infty} g_{\Lambda(a)}$ with probability one; because if we choose a box $\Lambda(b)$ with $b < a$, namely $b_i < a_i$ for each i, we then have the trivial estimate

$$\begin{aligned}|f_\Lambda - f_{\Lambda(a)}| \leq &\left|f_\Lambda - \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j}\right| + \left|\frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j} - g_{\Lambda(a)}\right| + |g_{\Lambda(a)} - g_{\Lambda(b)}| \\ &+ \left|g_{\Lambda(b)} - \frac{1}{N_b^-(\Lambda)} \sum_{j=1}^{N_b^-(\Lambda)} f_{\Lambda_j}\right| + \left|\frac{1}{N_b^-(\Lambda)} \sum_{j=1}^{N_b^-(\Lambda)} f_{\Lambda_j} - f_{\Lambda(a)}\right|\end{aligned}$$

where each term is less than or equal to $\epsilon/5$ for each $\epsilon > 0$ according to lemma (2.1) and our preceding assumptions when a, b, $N_a^-(\Lambda)$ and $N_b^-(\Lambda)$ are large enough; all these estimates clearly hold with probability one, proving thereby the existence of the Van Hove limit $\lim_{\Lambda\nearrow\mathbf{Z}^\nu} f_\Lambda$ with probability one. Sufficient conditions for the existence of $g_{\Lambda(a)}$ and of its limit when $a \to +\infty$ can be found using strong laws of large numbers in a way we shall now describe; we first prove a result ensuring the existence of Brout's free energy, namely the average of (2.2) over all impurity configurations, which is going to play an important role from now on.

Lemma 2.2. The matrix elements of $\mathbf{J}^0_{\Lambda_j}$ and $\boldsymbol{\sigma}^2_{\Lambda_j}$ being strictly finite for each j, the means (Brout's free energy)

$$f^0_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j}; \boldsymbol{\sigma}^2_{\Lambda_j}; \lambda) \equiv \langle f_{\Lambda_j}(\mathbf{J}_{\Lambda_j}; \lambda)\rangle_{\mathbf{J}_{\Lambda_j}} \tag{2.5}$$

and the variances

$$\tau^2_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j}; \boldsymbol{\sigma}^2_{\Lambda_j}; \lambda) \equiv \langle(f_{\Lambda_j}(\mathbf{J}_{\Lambda_j}; \lambda) - f^0_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j}; \boldsymbol{\sigma}^2_{\Lambda_j}; \lambda))^2\rangle_{\mathbf{J}_{\Lambda_j}} \tag{2.6}$$

exist and are finite.

Proof. We first perform a Taylor expansion around the mean matrix $\mathbf{J}^0_{\Lambda_j}$; we have

$$f_{\Lambda_j}(\mathbf{J}_{\Lambda_j};\lambda)=f_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\lambda)+\operatorname{grad} f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)\cdot(\mathbf{J}_{\Lambda_j}-\mathbf{J}^0_{\Lambda_j}) \tag{2.7}$$

where $\mathbf{U}_{\Lambda_j}=\mu\mathbf{J}_{\Lambda_j}+(1-\mu)\mathbf{J}^0_{\Lambda_j}$ with $\mu\in(0,1)$; therefore

$$\begin{aligned}|f^0_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda)|\leqslant\langle|f_{\Lambda_j}(\mathbf{J}_{\Lambda_j};\lambda)|\rangle_{\mathbf{J}_{\Lambda_j}}\leqslant|f_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\lambda)|\\ +\langle\|\operatorname{grad} f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)\|\cdot\|\mathbf{J}_{\Lambda_j}-\mathbf{J}^0_{\Lambda_j}\|\rangle_{\mathbf{J}_{\Lambda_j}}\end{aligned} \tag{2.8}$$

by elementary inequalities; now from the prototype Hamiltonian (2.1) we get

$$\frac{\partial f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)}{\partial U_{rr'}}=-|\Lambda_j|^{-1}\langle\boldsymbol{S}_r\cdot\boldsymbol{S}_{r'}\rangle_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)$$

for all $r, r'\in\Lambda_j$, where $\langle\ \rangle_{\Lambda_j}$ denotes the usual Gibbs state on Λ_j; consequently

$$\|\operatorname{grad} f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)\|=\left(\sum_{r,r'\in\Lambda_j}\left|\frac{\partial f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)}{\partial U_{rr'}}\right|^2\right)^{1/2}\leqslant 1. \tag{2.9}$$

On the other hand we have

$$\begin{aligned}\langle\|\mathbf{J}_{\Lambda_j}-\mathbf{J}^0_{\Lambda_j}\|\rangle_{\mathbf{J}_{\Lambda_j}}&\leqslant\langle\|\mathbf{J}_{\Lambda_j}-\mathbf{J}^0_{\Lambda_j}\|^2\rangle^{1/2}_{\mathbf{J}_{\Lambda_j}}\\ &=\left\langle\sum_{r,r'\in\Lambda_j}(J_{rr'}-J^0_{rr'})^2\right\rangle^{1/2}_{\mathbf{J}_{\Lambda_j}}=\left(\sum_{r,r'\in\Lambda_j}\sigma^2_{rr'}\right)^{1/2}\leqslant\sum_{r,r'\in\Lambda_j}\sigma_{rr'}.\end{aligned} \tag{2.10}$$

Substitution of (2.9) and (2.10) in (2.8) then leads to

$$|f^0_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda)|\leqslant|f_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\lambda)|+\sum_{r,r'\in\Lambda_j}\sigma_{rr'}<+\infty$$

which proves the first statement concerning (2.5). Now starting again from (2.7) we get

$$f^0_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda)=f_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\lambda)+\langle\operatorname{grad} f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)\cdot(\mathbf{J}_{\Lambda_j}-\mathbf{J}^0_{\Lambda_j})\rangle. \tag{2.11}$$

Therefore inserting (2.7) and (2.11) in (2.6), and using (2.9) again gives

$$\begin{aligned}\tau^2_{\Lambda_j}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda)&\leqslant\langle[\operatorname{grad} f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)\cdot(\mathbf{J}_{\Lambda_j}-\mathbf{J}^0_{\Lambda_j})]^2\rangle_{\mathbf{J}_{\Lambda_j}}\\ &\leqslant\langle\|\operatorname{grad} f_{\Lambda_j}(\mathbf{U}_{\Lambda_j};\lambda)\|^2\cdot\|\mathbf{J}_{\Lambda_j}-\mathbf{J}^0_{\Lambda_j}\|^2\rangle_{\mathbf{J}_{\Lambda_j}}\leqslant\sum_{r,r'\in\Lambda_j}\sigma^2_{rr'}<+\infty\end{aligned} \tag{2.12}$$

which proves the statement concerning (2.6). This achieves the proof.

Remarks. The preceding proof is based on the prototype Hamiltonian (2.1); no relevant modifications appear, however, when considering the various classical extensions and the quantum version of (2.1) with $D=3$ discussed at the very beginning of this section; all the basic estimates remain valid; that is true also for the various statements we are going to prove hereafter; we will therefore continue to consider (2.1) as the basis of our considerations in what follows. The first important connection between the random variable f_{Λ_j} and $f^0_{\Lambda_j}\equiv f_{\Lambda_j,\mathrm{q}}$ (quenched free energy) is proved in the following lemma.

Lemma 2.3. Suppose the matrix elements of $\boldsymbol{\sigma}_\Lambda^2$ are such that the quantity

$$M_\nu(\boldsymbol{\sigma}) = \sup_{r' \in \mathbb{Z}^\nu} \sum_{r \in \mathbb{Z}^\nu} \sigma^2_{r+r',r'} \tag{2.13}$$

exists and is finite. Then we have

$$\lim_{a\to+\infty} \lim_{N_a^-(\Lambda)\to+\infty} \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} (f_{\Lambda_j}(\mathbf{J}_{\Lambda_j}; \lambda) - f_{\Lambda_j,\mathrm{q}}(\mathbf{J}^0_{\Lambda_j}; \boldsymbol{\sigma}^2_{\Lambda_j}; \lambda)) = 0 \tag{2.14a}$$

with probability one.

Proof. The two-body couplings $J_{rr'}$ being independent random variables by assumption, the set $\{f_{\Lambda_j}\}_{j=1}^{N_a^-(\Lambda)}$ is a family of mutually independent random variables too since $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$; on the other hand we have, starting from (2.12),

$$\tau^2_{\Lambda_j} \leqslant \sum_{r,r' \in \Lambda_j} \sigma^2_{rr'} \leqslant \sum_{r' \in \Lambda_j} \sum_{r \in \mathbb{Z}^\nu} \sigma^2_{rr'} = \sum_{r' \in \Lambda_j} \sup_{r' \in \mathbb{Z}^\nu} \sum_{r \in \mathbb{Z}^\nu} \sigma^2_{r+r',r'} = |\Lambda_j| M_\nu(\boldsymbol{\sigma}) = |\Lambda(a)| M_\nu(\boldsymbol{\sigma})$$

which proves that

$$\sum_{j=1}^{\infty} \frac{\tau^2_{\Lambda_j}}{j^2} < +\infty;$$

therefore we have

$$\lim_{N_a^-(\Lambda)\to+\infty} \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} (f_{\Lambda_j}(\mathbf{J}_{\Lambda_j}; \lambda) - f_{\Lambda_j,\mathrm{q}}(\mathbf{J}^0_{\Lambda_j}; \boldsymbol{\sigma}^2_{\Lambda_j}; \lambda)) = 0 \tag{2.14b}$$

for each $a \in \mathbb{Z}^\nu$ with probability one by Kolmogorov's strong law of large numbers (Loève 1955, Renyi 1966, especially chap. 7). This proves (2.14a) with probability one.

Remarks. The preceding lemma shows that we have existence of $g_{\Lambda(a)}$, and of its limit when $a \to +\infty$ with probability one once we have existence of the corresponding quenched quantity

$$g_{\Lambda(a),\mathrm{q}} \equiv \lim_{N_a^-(\Lambda)\to+\infty} \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j,\mathrm{q}}$$

and of its limit when $a \to +\infty$. This last statement concerning $g_{\Lambda(a),\mathrm{q}}$ is in fact even sufficient to get the Van Hove limit of the quenched (Brout's) free energy $f_{\Lambda,\mathrm{q}}$, according to considerations similar to the ones given above relating to the Van Hove limit of f_Λ, and regarding the following lemma.

Lemma 2.4. Under the same conditions as in lemma 2.1 we have

$$\left| f_{\Lambda,\mathrm{q}}(\mathbf{J}^0_\Lambda; \boldsymbol{\sigma}^2_\Lambda; \lambda) - \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j,\mathrm{q}}(\mathbf{J}^0_{\Lambda_j}; \boldsymbol{\sigma}^2_{\Lambda_j}; \lambda) \right| \leqslant \epsilon \tag{2.15}$$

for any $\epsilon > 0$.

Proof. By (2.5) and (2.4) we get

$$\left|f_{\Lambda,q}(\mathbf{J}^0_\Lambda;\boldsymbol{\sigma}^2_\Lambda;\lambda)-\frac{1}{N_a^-(\Lambda)}\sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j,q}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda)\right|$$

$$\leqslant\left\langle\left|f_\Lambda(\mathbf{J}_\Lambda;\lambda)-\frac{1}{N_a^-(\Lambda)}\sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j}(\mathbf{J}_{\Lambda_j};\lambda)\right|\right\rangle_{\mathbf{J}_\Lambda}\leqslant\epsilon$$

which is (2.15).

Finally the basic inequality proving the equality with probability one in the thermodynamic limit of f_Λ and $f_{\Lambda,q}$ is established in the following lemma.

Lemma 2.5. Under the same conditions as in lemmas 2.1 and 2.3 we have, for any $\epsilon>0$ and every $\Lambda\subset\mathbb{Z}^\nu$ large enough in the sense of Van Hove,

$$|f_\Lambda(\mathbf{J}_\Lambda;\lambda)-f_{\Lambda,q}(\mathbf{J}^0_\Lambda;\boldsymbol{\sigma}^2_\Lambda;\lambda)|\leqslant\epsilon \tag{2.16}$$

with probability one.

Proof. We have

$$|f_\Lambda(\mathbf{J}_\Lambda;\lambda)-f_{\Lambda,q}(\mathbf{J}^0_\Lambda;\boldsymbol{\sigma}^2_\Lambda;\lambda|$$

$$\leqslant\left|f_\Lambda(\mathbf{J}_\Lambda;\lambda)-\frac{1}{N_a^-(\Lambda)}\sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j}(\mathbf{J}_{\Lambda_j};\lambda)\right|$$

$$+\left|\frac{1}{N_a^-(\Lambda)}\sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j}(\mathbf{J}_{\Lambda_j};\lambda)-\frac{1}{N_a^-(\Lambda)}\sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j,q}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda)\right|$$

$$+\left|\frac{1}{N_a^-(\Lambda)}\sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j,q}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda)-f_{\Lambda,q}(\mathbf{J}^0_{\Lambda,q};\boldsymbol{\sigma}^2_\Lambda;\lambda)\right|.$$

Now the first term on the right-hand side is less than $\epsilon/3$ with probability one according to lemma 2.1; the second is less than $\epsilon/3$ with probability one according to lemma 2.3, and the last term is less than $\epsilon/3$ according to lemma (2.4). This proves (2.16). We now summarise our results in the following theorem.

Theorem 2.6. Suppose the matrix elements of $\mathbf{J}_\Lambda$, $\mathbf{J}^0_\Lambda$ and $\boldsymbol{\sigma}^2_\Lambda$ obey the various conditions given above and assume that the limit

$$\lim_{a\to+\infty}\lim_{N_a^-(\Lambda)\to\infty}\frac{1}{N_a^-(\Lambda)}\sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j,q}(\mathbf{J}^0_{\Lambda_j};\boldsymbol{\sigma}^2_{\Lambda_j};\lambda) \tag{2.17}$$

exists and is finite. Then the Van Hove limit of the quenched (Brout's) free energy

$$f_q(\mathbf{J}^0;\boldsymbol{\sigma};\lambda)\equiv\lim_{\Lambda\nearrow\mathbf{Z}^\nu} f_{\Lambda,q}(\mathbf{J}^0_\Lambda;\boldsymbol{\sigma}^2_\Lambda;\lambda)$$

exists and is finite; furthermore the Van Hove limit of the random free energy

$$f(\mathbf{J};\lambda)\equiv\lim_{\Lambda\nearrow\mathbf{Z}^\nu} f_\Lambda(\mathbf{J};\lambda)$$

exists and is finite with probability one, and we have

$$f_q(\mathbf{J}^0;\boldsymbol{\sigma};\lambda)=f(\mathbf{J};\lambda)$$

with probability one.

Remarks and comments. Of course if condition (2.17) is not satisfied, the existence of f_q is not ensured any more; however, lemma (2.5) proves anyway that if one of the two Van Hove limits exists (with probability one for f_Λ) then the other one exists too and is equal to the former with probability one. Sufficient conditions implying existence of limits like (2.17) have been studied by Roos (1975); in the next section we are going to exhibit models for which (2.17) holds in a trivial way. On the other hand, observe that our arguments also show that Brout's prescription can only be expected to be valid in the limit of an infinitely large system, and still up to an exceptional set (of measure zero) of impurity configurations, and up to a sufficiently strong control on the randomness at large distances (relation (2.13)); this last condition is very natural in fact. Remark finally that the preceding considerations generalise those given by McCoy and Wu concerning the Ising lattice with random frozen impurities, based on the exact Onsager solution (McCoy 1970, McCoy and Wu 1971, McCoy 1972 and references therein). In the next section, we are going to consider random mean-field models to illustrate the preceding considerations.

3. An example: a class of random mean-field models

We still consider models defined by the Hamiltonian (2.1) or by its various extensions quoted above, but we assume that all the random couplings $J_{rr'}$ have the same mean, J^0, and the same variance, σ^2; in other words, we suppose that every probability distribution $P_{rr'}$ in

$$\langle\ \rangle_{J_\Lambda} = \prod_{r\neq r'\in\Lambda} \mathrm{d}J_{rr'} P_{rr'}(J_{rr'})$$

is independent of the corresponding bond (r, r'), namely $P_{rr'}(J_{rr'}) = P(J_{rr'})$ for each r, r'; important examples have already been given at the beginning of § 2 in connection with the spin glass problem. Those conditions define what we shall call, in some slightly generalised sense, a class of random mean-field models since such a situation could also be generated by considering the random Hamiltonian (2.1) with $J_{rr'} = J$ (supposed to be $O(|\Lambda|^{-1})$ when $|\Lambda| \to +\infty$) for all $r, r' \in \Lambda$; they will play, as we shall see, the role of a natural substitute for the usual translational invariance of the interactions. Of course when all the means and all the variances have the same value, condition (2.13) and convergence of the series ensuring the occurrence of the event $M_\nu(\mathbf{J}) < +\infty$ in lemma (2.1) (see the remark following that lemma) are not realised any more; however, it is still possible to use laws of large numbers to derive eventually a result similar to theorem 2.6. Suppose that the means of the random variables $J(r) = \sup_{r'\in\mathbf{Z}^\nu}|J_{r+r',r'}|$ exist and are finite. To be consistent with our preceding assumptions, suppose that these means are the same for each r. We then have the following result.

Lemma 3.1. Let J'^0 be the mean of $J(r) = \sup_{r'\in\mathbf{Z}^\nu}|J_{r+r',r'}|$ for each r; suppose that J'^0 is $O(|\Lambda|^{-1})$ when $|\Lambda| \to +\infty$; without loss of generality assume that $J'^0 = \tilde{J}^0/|\Lambda|$ where $\tilde{J}^0$ is finite and $O(1)$ when $|\Lambda| \to +\infty$. Then we have

$$\sum_{r\in\mathbf{Z}^\nu} J(r) = \sum_{r\in\mathbf{Z}^\nu} \sup_{r'\in\mathbf{Z}^\nu} |J_{r+r',r'}| < +\infty \tag{3.1}$$

with probability one.

Proof. We have $\Sigma_{r\in\Lambda} J'^0 = \bar{J}^0$ by assumption, thereby proving (3.1) by Beppo Levi's theorem.

Observe that the last statement is valid without any assumption on the corresponding variances.

Now we can prove a result similar to theorem (2.6). We have indeed the following theorem.

Theorem 3.2. Let J'^0 be $O(|\Lambda|^{-1})$ when $|\Lambda| \to +\infty$. Then for the class of random mean-field models defined above, the Van Hove limit of the quenched (Brout's) free energy

$$f_q(\mathbf{J}^0; \boldsymbol{\sigma}^0; \lambda) = \lim_{\Lambda \nearrow \mathbf{Z}^\nu} f_{\Lambda,q}(\mathbf{J}^0_\Lambda; \boldsymbol{\sigma}^2_\Lambda; \lambda)$$

exists and is finite; furthermore the Van Hove limit of the random free energy

$$f(\mathbf{J}; \lambda) = \lim_{\Lambda \nearrow \mathbf{Z}^\nu} f_\Lambda(\mathbf{J}_\Lambda; \lambda)$$

exists and is finite with probability one, and we have

$$f(\mathbf{J}; \lambda) = f_q(\mathbf{J}^0; \boldsymbol{\sigma}^2; \lambda) \tag{3.2}$$

with probability one.

Proof. From lemma (3.1) we get $M_\nu(\mathbf{J}) < +\infty$ with probability one so that inequality (2.4) holds with probability one; lemma (2.4) is then still valid without any change; but now we have $f_{\Lambda_j,q} = f_{\Lambda(a),q}$ for each j in (2.15), regarding our assumption concerning $\langle\ \rangle_{\mathbf{J}_{\Lambda_j}}$; that proves that f_q exists and is finite. Now lemma (2.2) is still valid as well, with $\tau^2_{\Lambda_j} = \tau^2_{\Lambda(a)}$ for each j, proving thereby that

$$\lim_{N_a^-(\Lambda) \to +\infty} \frac{1}{N_a^-(\Lambda)} \sum_{j=1}^{N_a^-(\Lambda)} f_{\Lambda_j}(\mathbf{J}_{\Lambda_j}; \lambda) = f_{\Lambda(a),q}(\mathbf{J}^0_{\Lambda(a)}; \boldsymbol{\sigma}^2_{\Lambda(a)}; \lambda)$$

with probability one; since this last relation plays the role of (2.14*b*) for random mean-field models, that proves, according to lemma (2.5), that $f(\mathbf{J}_\Lambda; \lambda)$ exists and is finite with probability one, and that (3.2) holds. This proves the theorem.

Remarks and comments. Observe first that the considered class of random mean-field models contains, to the best of our knowledge, almost every current model used to describe the spin glass phenomenon, in particular when a Gaussian distribution is chosen for the P. Our results are, however, independent of the particular form of the probability distribution; other distributions can therefore also be chosen as well, for example a Laplace distribution with mean J^0 and variance σ^2, namely

$$P(J_{rr'}) = \frac{\sqrt{2}}{2\sigma} \exp\left(\frac{-\sqrt{2}|J_{rr'} - J^0|}{\sigma}\right)$$

for $0 < \sigma < \sqrt{2}$, or a distribution for random ferromagnets, for example

$$P(J_{rr'}) = \begin{cases} 0 & \text{for } J_{rr'} < 0 \\ (J^0)^{-1} \exp\left(-\dfrac{J_{rr'}}{J^0}\right) & \text{for } J_{rr'} \geqslant 0 \end{cases}$$

with mean $J^0 > 0$ and variance $(J^0)^2$. We are not going to examine these various models further, since we think that the models studied in the last section are more interesting anyway, compared with random mean-field models where rather unphysical volume dependences have to be considered, like that of J'^0 in the preceding considerations, to get any sensible result. In the next section, we are going to propose similar considerations for some thermodynamic functions, and apply them to the existence problem of phase transitions in magnetic (spin) glasses.

4. Absence of mean random ordering in a class of one- and two-dimensional spin glasses

In this last section, the prototype Hamiltonian will still be (2.1), whose corresponding random free energy is given by (2.2). We are going to exhibit first a relation between the derivatives f'_Λ and $f'_{\Lambda,\mathrm{q}}$, with respect to λ, of the free energies f_Λ and $f_{\Lambda,\mathrm{q}}$ in the thermodynamic limit; the latter will be performed along the sequence of boxes

$$\Lambda_m \equiv \Lambda(a_m) = \{r \in \mathbb{Z}^\nu ; 0 \leq r^i < a^i_m ; i = 1, \ldots, \nu\}$$

where the sequence $\{a^i_m\}_{m\in\mathbb{N}}$ is increasing in m for each i. We then have the following theorem.

Theorem 4.1. Suppose the conditions of theorems (2.6) or (3.2) are satisfied. Then we have

$$f'(\mathbf{J}; \lambda) = f'_\mathrm{q}(\mathbf{J}^0; \boldsymbol{\sigma}^2; \lambda) \tag{4.1}$$

with probability one for all $\lambda \in \mathbb{R}$ with the exception of some possible countable set $\{\lambda_K\}_{K\in\mathbb{N}}$.

Proof. According to Fisher's theorem (Fisher 1965, especially appendix A) we have

$$f'_\mathrm{q}(\mathbf{J}^0; \boldsymbol{\sigma}^2; \lambda) = \lim_{m\to+\infty} f'_{\Lambda_m,\mathrm{q}}(\mathbf{J}^0_{\Lambda_m}; \boldsymbol{\sigma}^2_{\Lambda_m}; \lambda)$$

and

$$f'(\mathbf{J}; \lambda) = \lim_{m\to+\infty} f'(\mathbf{J}_{\Lambda_m}; \lambda)$$

(the latter with probability one) for all $\lambda \in \mathbb{R}$ with the exception of a possible countable set $\{\lambda_K\}_{K\in\mathbb{N}}$ of non-derivability points; now according to theorem (2.6) or (3.2) we have

$$f(\mathbf{J}; \lambda) = f_\mathrm{q}(\mathbf{J}^0; \boldsymbol{\sigma}^2; \lambda)$$

with probability one for all $\lambda \in \mathbb{R}$; that proves (4.1).

Remark. In fact, the last combination of Fisher's theorem and our theorems (2.6) and (3.2) proves more: it proves that, with probability one, the left-derivatives and the right-derivatives of $f(\mathbf{J}; \lambda)$ and $f_\mathrm{q}(\mathbf{J}^0; \boldsymbol{\sigma}^2; \lambda)$ are equal. We are now going to be much more specific applying similar ideas to the existence problem of phase transitions in one- and two-dimensional spin glasses. From now on we shall assume a positive external field $\lambda > 0$ in (2.1) and $D \geq 2$; in other words, the last condition means that we shall consider only models with continuous internal symmetry groups (SO(D), with

$D \geq 2$): our results do not apply to models with discrete internal symmetries like the one proposed by Sherrington and Kirkpatrick (1975). They have been announced elsewhere without (detailed) proofs (Vuillermot 1977). We shall assume that the three matrices $\mathbf{J}_\Lambda$, $\boldsymbol{\sigma}^2_\Lambda$ and $\mathbf{J}^0_\Lambda$, satisfying the assumptions of § 2, are in addition symmetric. Writing then $\langle\ \rangle_\Lambda(\mathbf{J}_\Lambda; \lambda)$ for the usual Gibbs state on Λ, we define the random magnetisation $m_\Lambda(\mathbf{J}_\Lambda; \lambda)$ by the derivative of (2.2) with respect to λ (up to a sign), getting

$$m_\Lambda(\mathbf{J}_\Lambda; \lambda) = |\Lambda|^{-1} \sum_{r\in\Lambda} \langle S^D_r\rangle_\Lambda(\mathbf{J}_\Lambda; \lambda). \tag{4.2}$$

And, for $2 \leq n \in \mathbb{N}$, we introduce a family of spin glass mean order parameters

$$q^{(n)}_\Lambda(\mathbf{J}_\Lambda; \lambda) = |\Lambda|^{-n}\Big(\sum_{r\in\Lambda} \langle S^D_r\rangle(\mathbf{J}_\Lambda; \lambda)\Big)^n. \tag{4.3}$$

Of course, regarding recent experimental facts (Murnick *et al* 1976), the right candidates to describe a *pure* spin glass phase should be defined by

$$\tilde{q}^{(n)}_\Lambda(\mathbf{J}_\Lambda; \lambda) = |\Lambda|^{-1} \sum_{r\in\Lambda} \langle S^D_r\rangle^n(\mathbf{J}_\Lambda; \lambda) \tag{4.4}$$

rather than by (4.3), with a special attention to

$$\tilde{q}^{(2)}_\Lambda(\mathbf{J}_\Lambda; \lambda) = |\Lambda|^{-1} \sum_{r\in\Lambda} \langle S^D_r\rangle^2(\mathbf{J}_\Lambda; \lambda).$$

However, we are going to discuss the existence of a one- and two-dimensional random ordering described by (4.3) rather than by (4.4), both because we have not yet found any useful (Bogoliubov-type) inequality to estimate the latter ones, and because the former ones are interesting anyway as describing a long-range order which can be considered as a *mixture* of a usual (ferro- or antiferro-) magnetic order and of a frozen-in (spin glass) order observed in some metallic alloys. The corresponding quenched quantities are then defined by

$$m_{\Lambda,q}(\mathbf{J}^0_\Lambda; \boldsymbol{\sigma}^2_\Lambda; \lambda) = -\frac{\mathrm{d}f_{\Lambda,q}}{\mathrm{d}\lambda} = |\Lambda|^{-1} \sum_{r\in\Lambda} \langle\langle S^D_r\rangle_\Lambda(\mathbf{J}_\Lambda; \lambda)\rangle_{\mathbf{J}_\Lambda} \tag{4.5}$$

and by

$$q^{(n)}_{\Lambda,q}(\mathbf{J}^0_\Lambda; \boldsymbol{\sigma}^2_\Lambda; \lambda) = |\Lambda|^{-n}\Big\langle\Big(\sum_{r\in\Lambda} \langle S^D_r\rangle(\mathbf{J}_\Lambda; \lambda)\Big)^n\Big\rangle_{\mathbf{J}_\Lambda}. \tag{4.6}$$

A first interesting relation between (4.2) and (4.3) is given in the following lemma.

Lemma 4.2. With $\lambda > 0$ we have for any $n \in \mathbb{N}$ and every impurity configuration, the inequalities

$$0 \leq \lim_{m\to+\infty} \inf q^{(n)}_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) \leq \lim_{m\to+\infty} \sup q^{(n)}_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) \leq \lim_{m\to+\infty} \sup m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) \leq 1. \tag{4.7}$$

Proof. For any impurity configuration we have $-\mathrm{d}^2 f_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda)/\mathrm{d}\lambda^2 \geq 0$ by concavity of f_{Λ_m} in λ (thermodynamical stability); (4.2) is then a monotone non-decreasing function of λ; therefore $m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) \geq 0$ whenever $\lambda > 0$ since $m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; 0) = 0$ by symmetry; furthermore, we have $m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) \leq 1$ from (4.2). Those facts then imply $0 \leq q^{(n)}_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) \leq 1$ whenever $\lambda > 0$; that proves (4.7).

Now consider the first Brillouin zone Δ_m corresponding to Λ_m, namely

$$\Delta_m = \{K \in \mathbb{R}^\nu;\ K^i = 2\pi n^i / a^i_m;\ n^i \in \mathbb{Z};\ -\tfrac{1}{2}a^i_m < n^i \leqslant \tfrac{1}{2}a^i_m;\ i = 1, \ldots, \nu\}$$

and suppose the random variable

$$N_\nu(\mathbf{J}) = \sum_{r \in \mathbf{Z}^\nu} r^2 \sup_{r' \in \mathbf{Z}^\nu} |J_{r+r',r'}| \tag{4.8}$$

is finite with probability one; then we can assume without loss that the potential defined from the $J_{rr'}$ is of strictly finite range, since every potential such that (4.8) holds can be approximated by finite-range potentials (Vuillermot and Romerio 1975, Ruelle 1972). We then have the following result.

Lemma 4.3. Under the conditions given above on the random matrix $\mathbf{J}_{\Lambda_m}$, there exists a constant $B > 0$ such that the inequality

$$1 \geqslant |\Lambda_m|^{-1} \sum_{K \in \Delta_m} \frac{m^2_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda)}{N_\nu(J)K^2 + \lambda B} \tag{4.9}$$

holds with probability one.

Proof. The reasoning is exactly the same as in lemma (2.1): for a fixed configuration of impurities such that $N_\nu(\mathbf{J})$ is finite, (4.9) follows from a slight modification of the original arguments by Mermin and Wagner (1966), Mermin (1967), and Vuillermot and Romerio (1975); here again, such a modification is necessary to take into account the absence of any translational invariance in the $J_{rr'}$. And since $N_\nu(\mathbf{J}) < +\infty$ holds with probability one, (4.9) also holds with probability one.

Now combination of the two preceding lemmas proves absence of random ordering in one and two dimensions, as follows.

Theorem 4.4. Under the same conditions as in lemma (4.3) we have, for any $n \in \mathbb{N}$:

$$\lim_{\lambda \downarrow 0} \liminf_{m \to +\infty} q^{(n)}_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) = \lim_{\lambda \downarrow 0} \limsup_{m \to +\infty} q^{(n)}_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) = 0 \tag{4.10}$$

with probability one, whenever $\nu \leqslant 2$.

Proof. From lemma (4.3) we get

$$\lim_{\lambda \downarrow 0} \limsup_{m \to +\infty} m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) = 0$$

with probability one, whenever $\nu \leqslant 2$. The statement (4.10) then follows from lemma (4.2).

Remark. Here again, convergence of

$$\sum_{r \in \mathbf{Z}^\nu} r^2 \sup_{r' \in \mathbf{Z}^\nu} |J_{r+r',r'}|$$

with probability one can be ensured by a condition on the means (Beppo Levi's theorem) or by a condition on the means and the variances (laws of large numbers). We shall not consider this problem further.

Now, as announced above, we are going to show the relation between the random quantities m_{Λ_m}, $q^{(n)}_{\Lambda_m}$ and the corresponding quenched parameters $m_{\Lambda_m,\mathrm{q}}$ and $q^{(n)}_{\Lambda_m,\mathrm{q}}$. The first ingredient we will need for that purpose is a *reversed* Hölder–Minkowski inequality written as follows.

Lemma 4.5. Let $(\Omega; \mu)$ be any measure space and let f, g be two positive functions in $L^1(\Omega; \mu)$ such that $\int_\Omega \mathrm{d}\mu(\omega)g(\omega)>0$. Then for any $p>1$ we have

$$\int_\Omega \mathrm{d}\mu(\omega)\frac{f^p(\omega)}{g^{p-1}(\omega)} \geqslant \frac{(\int_\Omega \mathrm{d}\mu(\omega)f(\omega))^p}{(\int_\Omega \mathrm{d}\mu(\omega)g(\omega)^{p-1}}. \tag{4.11}$$

Proof. If $\int_\Omega \mathrm{d}\mu(\omega)f(\omega)=0$, the inequality is trivial; suppose then $\int_\Omega \mathrm{d}\mu(\omega)f(\omega)>0$; in that way, from the theorem of arithmetic and geometric means (Hardy *et al* 1959) we get the inequality

$$\int \mathrm{d}\mu(\omega)f^p(\omega)g^q(\omega) \leqslant \left(\int \mathrm{d}\mu(\omega)f(\omega)\right)^p\left(\int \mathrm{d}\mu(\omega)g(\omega)\right)^q$$

whenever $p>0$, $q>0$ and $p+q=1$. A simple argument of Hardy *et al* (1959) then implies the *reversed* inequality

$$\int \mathrm{d}\mu(\omega)f^p(\omega)g^{1-p}(\omega) \geqslant \left(\int \mathrm{d}\mu(\omega)f(\omega)\right)^p\left(\int \mathrm{d}\mu(\omega)g(\omega)\right)^{1-p}$$

whenever $p>1$, which is equivalent to (4.11).

The second ingredient is a Bogoliubov inequality for the quenched quantity $m_{\Lambda_m,\mathrm{q}}$ derived as follows.

Lemma 4.6. Assume that the mean value, $N^0_\nu(\mathbf{J})$, of the random variable $N_\nu(\mathbf{J})$ exists, is strictly positive and finite: then we have

$$1 \geqslant |\Lambda_m|^{-1} \sum_{K\in\Delta_m} \frac{m^2_{\Lambda_m,\mathrm{q}}(\mathbf{J}^0_{\Lambda_m}; \boldsymbol{\sigma}^2_{\Lambda_m}; \lambda)}{N^0_\nu(\mathbf{J})K^2+\lambda B}. \tag{4.12}$$

Proof. We start from (4.9) and take its average with respect to $\langle\ \rangle\mathbf{J}_{\Lambda_m}$; we then apply inequality (4.11) with $\Omega \equiv \mathbb{R}^{|\Lambda_m|(|\Lambda_m|-1)/2}$ (the configuration space of impurities), $\mu \equiv \langle\ \rangle\mathbf{J}_{\Lambda_m}$, $f \equiv |m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda)|$, $g \equiv N_\nu(\mathbf{J})K^2+\lambda B$ and $p=2$; these successive steps give

$$\begin{aligned}1 &\geqslant |\Lambda_m|^{-1} \sum_{K\in\Delta_m} \left\langle \frac{|m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda)|^2}{N_\nu(\mathbf{J})K^2+\lambda B}\right\rangle_{\mathbf{J}_{\Lambda_m}} \geqslant |\Lambda_m|^{-1} \sum_{K\in\Delta_m} \frac{\langle |m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda)|\rangle^2_{\mathbf{J}_{\Lambda_m}}}{N^0_\nu(J)K^2+\lambda B}\\ &\geqslant |\Lambda_m|^{-1} \sum_{K\in\Delta_m} \frac{|\langle m_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda)\rangle_{\mathbf{J}_{\Lambda_m}}|^2}{N^0_\nu(\mathbf{J})K^2+\lambda B} = |\Lambda_m|^{-1} \sum_{K\in\Delta_m} \frac{m^2_{\Lambda_m,\mathrm{q}}(\mathbf{J}^0_{\Lambda_m}; \boldsymbol{\sigma}^2_{\Lambda_m}; \lambda)}{N^0_\nu(\mathbf{J})K^2+\lambda B}\end{aligned}$$

which achieves the proof.

The last useful ingredient is the following lemma.

Lemma 4.7. With $\lambda>0$ we have for any $n\in\mathbb{N}$ the inequalities

$$0 \leqslant \liminf_{m\to+\infty} q^{(n)}_{\Lambda_m,\mathrm{q}} \leqslant \limsup_{m\to+\infty} q^{(n)}_{\Lambda_m,\mathrm{q}} \leqslant \limsup_{m\to\infty} m_{\Lambda_m,\mathrm{q}} \leqslant 1. \tag{4.13}$$

Proof. As for lemma (4.2) according to (4.5) and (4.6).

We then have the following statement for the quenched order parameters.

Theorem 4.8. Under the same conditions as in lemma 4.6 we have, for any $n \in \mathbb{N}$:

$$\lim_{\lambda \downarrow 0} \liminf_{m \to \infty} q^{(n)}_{\Lambda_m,\mathrm{q}}(\mathbf{J}^0_{\Lambda_m}; \boldsymbol{\sigma}^2_{\Lambda_m}; \lambda) = \lim_{\lambda \downarrow 0} \limsup_{m \to +\infty} q^{(n)}_{\Lambda_m,\mathrm{q}}(\mathbf{J}^0_{\Lambda_m}; \boldsymbol{\sigma}^2_{\Lambda_m}; \lambda) = 0 \tag{4.14}$$

whenever $\nu \leq 2$.

Proof. As for theorem (4.4), we use (4.12) and (4.13).

Remarks. As a trivial corollary of theorems (4.4) and (4.8) we have for any $n \in \mathbb{N}$, the inequality

$$\lim_{\lambda \downarrow 0} \liminf_{m \to +\infty} q^{(n)}_{\Lambda_m}(\mathbf{J}_{\Lambda_m}; \lambda) = \lim_{\lambda \downarrow 0} \limsup_{m \to \infty} q^{(n)}_{\Lambda_m,\mathrm{q}}(\mathbf{J}^0_{\Lambda_m}; \sigma^2_{\Lambda_m}; \lambda)$$

with probability one, whenever $\nu \leq 2$, which shows with theorem 4.1, that the equalities between the free energies f and f_q proven in the last two sections persist, in some sense, for some of the corresponding thermodynamic functions; similar statements can also be given, of course, for thermodynamic functions related to higher-order derivatives of the free energy.

5. Remarks, comments and open problems

We have shown in a rigorous way how Brout's free energy is related to the corresponding random free energy, in the thermodynamical limit, when the control on the randomness is strong enough. That constitutes an *a posteriori* justification of Brout's procedure for a class of infinite random systems at least, including the most current ones which are supposed to describe the spin glass phenomenon in a reasonable way. When the control on the randomness is not strong enough, however, there is no serious reason to replace f by f_q, and f is the right quantity we have to deal with, as in the two-dimensional theory of random Ising models proposed by McCoy and Wu. And even in the case where theorems (2.6) or (3.2) are valid, the equality between f and f_q is only true for almost all impurity configurations. Such a precise statement, though rather mathematical, should be considered very seriously by physicists: the fact that an exceptional set of impurity configurations for which $f \neq f_\mathrm{q}$ may exist is certainly responsible for dramatic changes in the critical properties of the corresponding systems, as it is the case for the McCoy and Wu model; in a model where f would be equal to f_q everywhere, the notion of randomness would be useless, the critical properties similar to those of the non-random model described by f_q, and the ultimate comparison with experiments presumably very poor and disappointing. Observe also that, whereas thermodynamical quantities are probability-one objects in the preceding sense, the same statement does not persist any more for non-thermodynamical quantities like the spin–spin correlations functions; the McCoy and Wu model once more gives an explicit example of that situation. Throughout we have dealt with independent random variables, but the extension of our results to the dependent case is an interesting open problem. Likewise, a serious discussion of the quantities (4.4) rather than (4.3) would be useful.

Acknowledgments

The author would like to thank Professor P W Anderson, Professor E Lieb, Dr D C Licciardello, and Dr R Palmer for discussions, the Departments of Mathematics and Physics of Princeton University for their kind hospitality, and the Swiss National Science Foundation for Scientific Research for having supported him financially under Grant 820.436.76.

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